

On pressure boundary conditions for thermoconvective problems

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SUMMARY

Solving numerically hydrodynamical problems of incompressible fluids raises the question of handling first order derivatives (those of pressure) in a closed container and determining its boundary conditions. A way to avoid the first point is to derive a Poisson equation for pressure, although the problem of taking the right boundary conditions still remains. To remove this problem another formulation of the problem has been used consisting of projecting the master equations into the space of divergence-free velocity fields, so pressure is eliminated from the equations. This technique raises the order of the differential equations and additional boundary conditions may be required. High-order derivatives are sometimes troublesome, specially in cylindrical coordinates due to the singularity at the origin, so for these problems a low order formulation is very convenient. We research several pressure boundary conditions for the primitive variables formulation of thermoconvective problems. In particular we study the Marangoni instability of an infinite fluid layer and we show that the numerical results with a Chebyshev collocation method are highly correspondent to the exact ones. These ideas have been applied to linear stability analysis of the Bénard–Marangoni (BM) problem in cylindrical geometry and the results obtained have been very accurate. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: boundary conditions; cylindrical geometry; Marangoni convection; Bénard–Marangoni convection; collocation method

1. INTRODUCTION

The problem of thermoconvective instabilities in fluid layers heated from below has become a classical subject in fluid mechanics [1, 2]. It is well known that two different effects are responsible for the onset of motion when the temperature difference becomes larger than a certain threshold: gravity and capillary forces. When both effects are taken into account the problem is called Bénard–Marangoni (BM) convection [3]. At the beginning theoretical studies considered layers of infinite horizontal extent without modelling lateral side-wall effects always present in experiments. More recently, experiments have been concerned with confined

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containers with different geometries [4–7], and consequently theoretical and numerical studies have followed the same way [8–14].

Solving numerically hydrodynamical problems of incompressible fluids raises the question of handling first order derivatives for pressure in a closed container and determining its boundary conditions. In the velocity–pressure formulation [15–17] first order derivatives are avoided by deriving a Poisson equation for pressure [15], although the problem of finding its boundary conditions still remains. Dirichlet and Neumann conditions are studied in detail in References [15] and [17] and good results are obtained with Neumann conditions in Reference [18]. In this work, sensitivity of the results with respect to artificial pressure boundary conditions is reported. The results we mention refer to rigid boundary conditions in the velocity field and purely hydrodynamical problems, but the issue of thermoconvection with free or Marangoni conditions for the velocity at the upper surface which we treat here has not been addressed. To avoid spurious results in the primitive variables formulation, finite difference and finite element methods use staggered grids, but this is not necessary with the collocation method we propose in this article.

In thermoconvection problems pressure is usually avoided. In References [19] and [20] the method of potentials of velocity is used to eliminate the variable of pressure from the equations. This technique raises the order of the differential equations and additional boundary conditions may be required. This is particularly troublesome in cylindrical coordinates where high-order derivatives cause awkward difficulties. The spectral method used in Reference [14] allows the removal of pressure in the primitive variables formulation, however the collocation method that we study here is easier to implement.

In this paper we present several pressure boundary conditions that allow us to solve thermoconvective problems in the primitive variables formulation. This is shown to be very useful when dealing with cylindrical coordinates. In the second section we perform the linear stability analysis of the Marangoni problem in an infinite horizontal fluid layer. Since it has an exact solution it is used as a test problem to be compared with the numerical results obtained in the primitive variables formulation and several pressure boundary conditions. We use a Chebyshev collocation method and sensitivity to the pressure boundary conditions is observed only in the convergence rate of the method. In the third section we study the BM problem in cylindrical geometry in the primitive variables formulation with the boundary conditions that provide better convergence for the test problem. We explain the methodology in detail. We study the convergence of the numerical results and compare them with the previous bibliography on the subject. In the fourth section conclusions are presented.

2. TEST PROBLEM

The physical set-up we consider corresponds to an infinite layer of a fluid with a free top surface. The domain is:

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 / (x, y) \in \mathbb{R}^2, 0 < z < 1\} \quad (1)$$

In the vertical direction a temperature gradient is imposed. It is well known that motion sets in after the vertical temperature gradient has reached a critical value. The only mechanism that we consider in developing convection is the Marangoni effect, which consists of the variation of the surface tension with temperature. In the reference state there is no motion and heat

propagates by conduction only. To study the linear stability of this state we must solve the linearized equations for the perturbations from the reference state. As there are no boundary conditions in the horizontal directions and the basic state is invariant under translations in those directions, it is possible to take the Fourier modes in those coordinates,

$$\mathbf{u}' = \mathbf{u}(z)e^{i\mathbf{k}\mathbf{x}} + cc., \quad \theta' = \theta(z)e^{i\mathbf{k}\mathbf{x}} + cc., \quad p' = p(z)e^{i\mathbf{k}\mathbf{x}} + cc \tag{2}$$

The linear problem to be solved is

$$\nabla \cdot \mathbf{u} = 0 \tag{3}$$

$$Pr^{-1} \partial_t \mathbf{u} = -\nabla p + \Delta \mathbf{u} \tag{4}$$

$$\partial_t \theta = u_z + \Delta \theta \tag{5}$$

where $\mathbf{u} = (u_x, u_y, u_z)$ is the velocity field of the fluid, θ is the temperature, p the pressure, $\nabla = (ik_x, ik_y, \partial_z)$, $\Delta = (\partial_z^2 - k^2)$, $k = |\mathbf{k}|$, Pr is the Prandtl number which is considered infinite. The boundary conditions are:

$$\mathbf{u}|_{z=0} = 0, \quad \theta|_{z=0} = 0 \tag{6}$$

$$(\partial_z u_x + Mik_x \theta)|_{z=1} = 0, \quad (\partial_z u_y + Mik_y \theta)|_{z=1} = 0 \tag{7}$$

$$u_z|_{z=1} = 0, \quad (\partial_z \theta + B\theta)|_{z=1} = 0 \tag{8}$$

where B is the Biot number and M is the Marangoni number which takes into account the variation of the surface tension with temperature.

2.1. Formulation without pressure

Classically, pressure is eliminated and the following set of equations and boundary conditions are obtained,

$$(D^2 - k^2)^2 u_z = 0 \tag{9}$$

$$u_z + (D^2 - k^2)\theta = 0 \tag{10}$$

$$u_z|_{z=0} = Du_z|_{z=0} = \theta|_{z=0} = 0 \tag{11}$$

$$u_z|_{z=1} = (D^2 u_z + k^2 M \theta)|_{z=1} = (D\theta + B\theta)|_{z=1} = 0 \tag{12}$$

where $D = \partial/\partial z$. This formulation will be called P1 in the following text. It can be easily shown that the u_z and θ solutions are

$$u_z(z) = a_1 e^{kz} + a_2 z e^{kz} + a_3 e^{-kz} + a_4 z e^{-kz} \tag{13}$$

$$\theta(z) = b_1 e^{kz} + b_2 e^{-kz} - \frac{1}{2} \frac{a_1 z e^{kz}}{k} + \frac{1}{2} \frac{a_3 z e^{-kz}}{k}$$

$$+ a_2 \left(\frac{1}{4} \frac{ze^{kz}}{k^2} - \frac{1}{4} \frac{z^2 e^{kz}}{k} \right) + a_4 \left(\frac{1}{4} \frac{ze^{-kz}}{k^2} + \frac{1}{4} \frac{z^2 e^{-kz}}{k} \right) \quad (14)$$

Introducing them into Equations (9)–(12) we get a linear system of equations for the unknowns a_i, b_i . The solvability condition for the system gives the following dependence of the critical Marangoni number on k ,

$$M = \frac{8kB(\alpha + 4\alpha k + \alpha^2 - 4\alpha^2 k - 1 - \alpha^3) + 8k^2(-1 - \alpha + 4k\alpha + \alpha^2 + 4k\alpha^2 + \alpha^3)}{-3\alpha^2 + 4k^3\alpha + 3\alpha + 4k^3\alpha^2 - 1 + \alpha^3}$$

where α is a function of k , $\alpha(k) = \cosh(2k) - \sinh(2k)$.

2.2. Formulation in primitive variables

We consider the primitive variables formulation with two different boundary conditions for pressure.

1. Continuity equation at the boundaries

$$(\nabla \cdot \mathbf{u})|_{z=0,1} = 0 \quad (15)$$

these boundary conditions together with Equations (3)–(5) and boundary conditions (6)–(8) define the problem P2.

2. Normal component of the Navier–Stokes equations at the top boundary and continuity equation at the bottom,

$$(-\partial_z p + \Delta u_z)|_{z=1} = 0 \quad (16)$$

$$(\nabla \cdot \mathbf{u})|_{z=0} = 0 \quad (17)$$

these boundary conditions together with Equations (3)–(5) and boundary conditions (6)–(8) define the problem P3.

2.3. Numerical method

The exact formulation (P1) together with formulations P2 and P3 are solved numerically with a Chebyshev collocation method. The eigenfunctions are approximated by Chebyshev polynomial expansions in the z direction. After changing the z coordinate to transform the $[0, 1]$ interval into $[-1, 1]$ we use the expansions,

$$\mathbf{u} = \sum_{n=0}^{N-1} \mathbf{a}_n T_n(z), \quad \theta = \sum_{n=0}^{N-1} b_n T_n(z), \quad p = \sum_{n=0}^{N-1} c_n T_n(z) \quad (18)$$

which are introduced into the equations.

For problem P1 the system and boundary conditions are evaluated at the collocation points.

$$z_i = \cos \left(\left(\frac{i-1}{N-1} - 1 \right) \pi \right), \quad i = 1, \dots, N \quad (19)$$

In particular, Equation (9) is evaluated at nodes $i = 3, \dots, N - 2$; Equation (10) at $i = 2, \dots, N - 1$; the first boundary conditions (11) at $i = N$ and the second ones (12) at $i = 1$. We obtain

Table I. Critical Marangoni number M_c for different orders of expansions and $B = 10, k = 10$.

N	P1	P2	P3
17	1600.01	1600.01	1600.01
15	1599.99	1599.91	1600.00
13	1599.42	1597.95	1599.66
11	1590.73	1572.61	1595.49
9	1500.53	1382.53	1572.57
7	972.58	776.62	1622.50
0	231.32	201.84	2817.82

Table II. Critical Marangoni number M_c and wave number k_c for different values of the Biot number B .

B	P1 (numerical)		P1 (exact)		P2		P3	
	M_c	k_c	M_c	k_c	M_c	k_c	M_c	k_c
0.1	83.20	2.04	83.43	2.03	82.93	2.05	83.31	2.03
1	115.69	2.26	116.13	2.25	115.14	2.28	115.92	2.25
10	410.45	2.77	413.44	2.74	406.27	2.82	412.20	2.75

$2 \times N$ unknowns and $2 \times N$ equations. For problems P2 and P3: (3) at the nodes $i = 2, \dots, N - 1$; (4) at $i = 2, \dots, N - 1$; (5) at $i = 2, \dots, N - 1$; the first boundary conditions (6) at $i = N$; (7) and (8) at $i = 1$. We obtain $5 \times N$ unknowns and $5 \times (N - 2) + 8$ equations. To complete the system we evaluate Equation (15) at $i = 1, N$ for problem P2, and Equation (16) at $i = 1$ and (17) at $i = N$ for problem P3. If the coefficients of the unknowns which form the matrices of the resulting linear system A and B satisfy $\det(A - \lambda B) = 0$, a nontrivial solution of the linear homogeneous system exists. This condition generates a dispersion relation $\lambda \equiv \lambda(k, M, B)$, equivalent to a direct calculation of the eigenvalues from the system $AX = \lambda BX$, where X is the vector which contains the unknowns. When λ becomes positive the basic state is unstable. In the critical situation $\lambda = 0$ with no imaginary part. In this case the dispersion relation can be written as $M \equiv M(k, B)$. We have calculated different critical Marangoni numbers for several k and B values.

2.4. Convergence of the numerical method

To carry out a test of the convergence of the method we compare the differences in the thresholds of M_c for different orders of expansions. In Table I the thresholds for $k = 10$ and $B = 10$ are shown for seven odd consecutive expansions. The thresholds converge to the exact value of 1600.01 and the difference between consecutive expansions tends to zero as N increases. Results on convergence improves greatly when k and B are smaller, i.e. for the critical value of k, k_c as Table II shows, for this reason there and in Table III we use seven polynomials expansions.

2.5. Discussion

As the exact solution is known, it is straightforward to compare the numerical methods. In Table II we show the critical Marangoni and wave numbers in each formulation for different

Table III. Critical Marangoni number M_c for different values of the Biot number B and $k = 10$.

B	P1 (numerical)	P1 (exact)	P2	P3
0.1	484.39	808.01	386.80	808.08
1	528.77	880.01	422.23	882.12
10	972.58	1600.01	776.62	1622.50

Biot numbers. We calculate differences between each formulation and the exact solution. The mean relative error for P1 and P2 is 10^{-2} , whereas the same for P3 is 2×10^{-3} . This indicates P3 as the best approach. In Table III we show the critical Marangoni number for $k = 10$ in each formulation and different Biot numbers. Although convergence has not been reached (see Table I) the result is significant because while the mean relative error for P1 is 0.5 and for P2 it is 0.4, for P3 it is 0.005. We confirm again that the third formulation converges better. Differences in accuracy between P1 and P3 could be due to the fact that P1 is a higher order problem. On the other hand P2 diminishes its efficiency with respect to P3 because of the boundary conditions considered. Although sensitivity of the numerical method to pressure boundary conditions has been addressed [18, 19] it seems here that it is not much more important than other effects such as increasing the order of the derivatives in the equations. Moreover, we have shown that considering appropriate boundary conditions for pressure leads to very good numerical performance.

3. BÉNARD–MARANGONI IN CYLINDRICAL GEOMETRY

3.1. Formulation of the problem

The physical set-up corresponds to a fluid layer of thickness d filling a cylinder with radius l . The surface tension at the upper free surface is temperature dependent and the fluid is heated from below. Motion sets in when the vertical temperature gradient has exceeded a critical value.

In the basic state, there is no motion and heat propagates by conduction. In the linear Boussinesq equations, the field can be expanded in the azimuthal variable ϕ in Fourier modes as follows,

$$\mathbf{u}(t, r, \phi, z) = \mathbf{u}_m(t, r, z)e^{im\phi} \quad (20)$$

$$\theta(t, r, \phi, z) = \theta_m(t, r, z)e^{im\phi} \quad (21)$$

$$p(t, r, \phi, z) = p_m(t, r, z)e^{im\phi} \quad (22)$$

where m is the wave number, θ and $\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\phi + w\mathbf{e}_z$ are the infinitesimal temperature and velocity perturbations with respect to the conductive solution, p is the pressure perturbation, and (r, ϕ, z) are the polar coordinates. If space, time, velocity, temperature and pressure fields are respectively divided by the constants d , d^2/κ , κ/d^2 , ΔT and $\rho_0\kappa\nu/d^2$, the equations in dimensionless form are obtained. Here κ is the thermal diffusivity, ρ_0 is the mean density, η is the dynamic viscosity (related to the kinematic viscosity through the expression, $\nu = \eta/\rho_0$)

and ΔT is the (conductive) temperature drop between the bottom and the top of the layer. The linearized general equations for the perturbations of index m are easily shown to be

$$Pr^{-1} \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial r} + \nabla_{m^2+1}^2 u - \frac{2im}{r^2} v \tag{23}$$

$$Pr^{-1} \frac{\partial v}{\partial t} = -\frac{im}{r} p + \nabla_{m^2+1}^2 v + \frac{2im}{r^2} u \tag{24}$$

$$Pr^{-1} \frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} + \nabla_m^2 w + R\theta \tag{25}$$

$$\frac{\partial \theta}{\partial t} = w + \nabla_m^2 \theta \tag{26}$$

$$0 = \frac{1}{r} \frac{\partial(ru)}{\partial r} + \frac{im}{r} v + \frac{\partial w}{\partial z} \tag{27}$$

where the subindex m has been cancelled for simplicity. The Rayleigh number is defined by $R = \alpha g \Delta T d^3 / \kappa \mu$, where α is the coefficient of volume expansion and g is the gravity. The Prandtl number is given by $Pr = \nu / \kappa$. ∇_n^2 is defined by

$$\nabla_n^2 = r^{-1} \partial / \partial r (r \partial / \partial r) - nr^{-2} + \partial^2 / \partial z^2$$

The boundary conditions are the following. The bottom of the box is rigid and assumed to be perfectly heat conducting so that

$$\mathbf{u} = 0, \quad \theta = 0, \quad \frac{1}{r} \frac{\partial(ru)}{\partial r} + \frac{im}{r} v + \frac{\partial w}{\partial z} = 0 \quad \text{on } z = 0 \tag{28}$$

The upper surface of the fluid is assumed to be plane, nondeformable and free where surface tension effects are taken into account. We also assume that at the top, heat is transferred from the liquid to the ambient gas according to Newton's law of cooling, which results in a Biot condition for the temperature perturbations. For pressure the normal projection of the Navier–Stokes equations into the top plane is considered as a boundary condition. The mathematical expressions of the boundary conditions at the upper surface are then,

$$w = 0, \quad \frac{\partial \theta}{\partial z} + B\theta = 0, \quad \frac{\partial u}{\partial z} + M \frac{\partial \theta}{\partial r} = 0 \quad \frac{\partial v}{\partial z} + M \frac{im}{r} \theta = 0 \quad \text{on } z = 1 \tag{29}$$

$$-\frac{\partial p}{\partial z} + \nabla_m^2 w + R\theta = 0 \quad \text{on } z = 1 \tag{30}$$

where B is the Biot number and $M = \gamma \Delta T d / \rho_0 \kappa \nu$ is the Marangoni number with γ the constant rate of change of surface tension with temperature.

The lateral side wall is rigid and we will consider it adiabatically insulated. For pressure we consider the normal projection of the Navier–Stokes equations into this wall as boundary condition. This is written as follows

$$\mathbf{u} = \mathbf{0}, \quad \partial \theta / \partial r = 0 \quad \text{on } r = a \tag{31}$$

$$-\frac{\partial p}{\partial r} + \nabla_{m^2+1}^2 u - \frac{2im}{r^2} v = 0 \quad \text{on } r = a \quad (32)$$

where $a = l/d$ is the aspect ratio.

The use of cylindrical coordinates, which are singular at $r = 0$, imposes regularity conditions on the velocity and temperature fields [22]. These conditions, which express that the unknown fields are single valued at $r = 0$, are mathematically summarized as follows,

$$\frac{\partial \mathbf{x}}{\partial \theta} = 0 \quad (33)$$

where \mathbf{x} is any scalar or vectorial field. For the temperature and velocity perturbations, these conditions are written as

$$u = \frac{\partial w}{\partial r} = \frac{\partial \theta}{\partial r} = \frac{\partial p}{\partial r} = 0, \quad m = 0 \quad (34)$$

$$u + iv = w = \theta = p = 0, \quad m = 1 \quad (35)$$

$$u = v = w = \theta = p = 0, \quad m \neq 0, 1 \quad (36)$$

3.2. Numerical method

The numerical technique used is a collocation method which is similar to that of Reference [23] for rectangular containers. Firstly we change the variables z and r to transform the intervals $[0, 1] \times [0, a]$ into $[-1, 1] \times [-1, 1]$ to use the Chebyshev polynomials. The fields in Equations (23)–(27), denoted generically by \mathbf{x} , are approximated by Chebyshev expansions,

$$\mathbf{x} = \sum_{l=0}^{L-1} \sum_{n=0}^{M-1} a_{ln}^{\mathbf{x}} T_l(r) T_n(z) \quad (37)$$

which are introduced into the equations. The resulting expressions are evaluated at the collocation points (r_j, z_i) defined as follows,

$$r_j = \cos \left(\left(\frac{j-1}{L-1} - 1 \right) \pi \right), \quad j = 1, \dots, L \quad (38)$$

$$z_i = \cos \left(\left(\frac{i-1}{N-1} - 1 \right) \pi \right), \quad i = 1, \dots, N \quad (39)$$

In the case $m > 1$ the system and boundary conditions are evaluated at the following collocation points: Equations (23)–(27) at the nodes $i = 2, \dots, N-1$, $j = 2, \dots, L-1$; the boundary conditions at $z = -1$ (28) at $i = 1$, $j = 2, \dots, L-1$; the boundary conditions at $z = 1$ (29) at $i = N$, $j = 2, \dots, L-1$ and (30) at $i = N$, $j = 2, \dots, L$; the boundaries at $r = -1$ (36) at $i = 1, \dots, N$, $j = 1$; finally the boundaries at $r = 1$ (31) at $i = 1, \dots, N$, $j = L$ and (32) at $i = 1, \dots, N-1$, $j = L$. We obtain $5 \times N \times L$ equations and $5 \times N \times L$ unknowns. For $m = 1$

Table IV. Critical Marangoni number M_c for different orders of expansions and $B = 2, a = 5$.

m	5×9	7×11	9×13	11×15
$m = 0$	150.705	153.689	154.064	154.062
$m = 1$	148.156	152.558	152.945	152.949
$m = 2$	150.946	153.734	154.154	154.154
$m = 3$	148.526	152.899	153.255	153.256
$m = 4$	149.400	153.542	153.979	153.987

the evaluation has been done at the same nodes, but as there are only four boundary conditions at $r = -1$ we have diminished the expansion range of the v field by one order,

$$v = \sum_{l=0}^{L-2} \sum_{n=0}^{N-1} a_{ln}^v T_l(r) T_n(z) \quad (40)$$

Therefore we get $4 \times N \times L + N \times (L - 1)$ equations with the same amount of unknowns. In the case $m = 0$ the angular component of the velocity is nulle and Equation (24), together with the corresponding boundary condition for r in Equations (28)–(36) disappear and therefore we obtain $4 \times N \times L$ equations with the same amount of unknowns. In this case the matrix associated to the linear algebraic system is singular, due to the fact that pressure is defined up to an additive constant. To fix this constant in the node $i = N - 2, j = L$ the boundary condition (32) is replaced by a Dirichlet condition for pressure (i.e., $p = 0$ at $i = N - 2, j = L$). The resulting linear systems have been solved with a standard numerical package. The eigenfunctions and thresholds of the generalized problem are numerically calculated. If the coefficients of the unknowns which form the matrices A and C satisfy $\det(A - \lambda C) = 0$, a nontrivial solution of the linear homogeneous system exists. This condition generates a dispersion relation $\lambda = \lambda(R, M, B)$, equivalent to a direct calculation of the eigenvalues from the system $AX = \lambda CX$, where X is the vector which contains the unknowns. When λ becomes positive the basic state is unstable. In the critical situation $Re(\lambda) = 0$ without imaginary part. In this case the dispersion relation can be written as $M \equiv M(R, B)$ or $R \equiv R(M, B)$ and critical Marangoni number or critical Rayleigh number are obtained directly from the eigenvalue problem.

3.3. Convergence of the numerical method

To carry out a test of the convergence of the method we compare the differences in the critical Marangoni number M_c to different orders of expansions for different values of the aspect ratio a . In Table IV these thresholds are shown for four consecutive expansions varying the number of polynomials taken in the z direction (N) and in the r direction (L). These results allow us to conclude that $N \times L = 9 \times 13$ gives very good results for aspect ratios lower than 5. So, except where otherwise stated, all results given below correspond to the values $N \times L = 9 \times 13$.

3.4. Comparison with other theoretical works

Our main goal is to show how a Chebyshev collocation method applied to the primitive variables formulation of thermoconvective problems with convenient boundary conditions for pressure produces excellent results.

Table V. Comparison between our results and those of Dauby *et al.* [14]. The first lines give the critical Marangoni number when $R = 0$ and for different Biot numbers: the bottom of the table presents the critical Rayleigh number for $M = 0$.

	$a = 1$		$a = 2$		$a = 4$	
$B(R=0)$	M_c (this paper)	M_c [14]	M_c (this paper)	M_c [14]	M_c (this paper)	M_c [14]
0.01	164.65	164.55	84.640	84.638	82.485	82.486
0.1	168.44	168.34	88.346	88.344	86.263	86.263
1	206.29	206.16	125.011	125.000	120.624	120.630
	R_c (this paper)	R_c [14]	R_c (this paper)	R_c [14]	R_c (this paper)	R_c [14]
0.01	1419.30	1419.47	712.542	712.667	695.543	695.668
0.1	1426.06	1426.24	726.574	726.704	709.326	709.457
1	1481.89	1482.12	835.897	836.072	799.522	799.735

Table VI. Critical Marangoni numbers M_c for different azimuthal wave number m . The Rayleigh and Biot numbers are 100 and 0.2, respectively.

	$a = 1$	$a = 2$	$a = 4$	$a = 8$
$m = 0$	163.676	80.878	78.777	76.179
$m = 1$	108.383	91.254	77.864	76.448
$m = 2$	158.994	98.407	79.699	76.203
$m = 3$	255.885	99.955	78.095	76.487

Dauby *et al.* [14] have considered the comparison between their results and the preceding ones. For this reason we are focusing on a comparison only with their paper. The correspondence between our work and that of Dauby *et al.* [14] can be tested by comparing our Tables V and VI to their Tables II and III. From this comparison we see that the deviations are less than 0.06 per cent. We have checked that the succession of unstable eigenmodes when a is increasing in our approach coincides with those of Dauby *et al.* [14]. In their work they discuss differences obtained with Zaman and Narayanan [13]. As is shown in Figure 1 when a increases all the critical values converge to the same thresholds and it is harder to distinguish among them with a numerical method.

4. CONCLUSIONS

We have shown that a Chebyshev collocation method applied to the primitive variables formulation of thermoconvective problems with convenient boundary conditions for pressure produces excellent results. First, we have solved the Marangoni problem in infinite geometry with a Chebyshev collocation method, keeping the primitive variables formulation and testing two sets of boundary conditions for pressure. We have compared these results with the obtained classically by eliminating pressure. As there is an exact solution to this problem

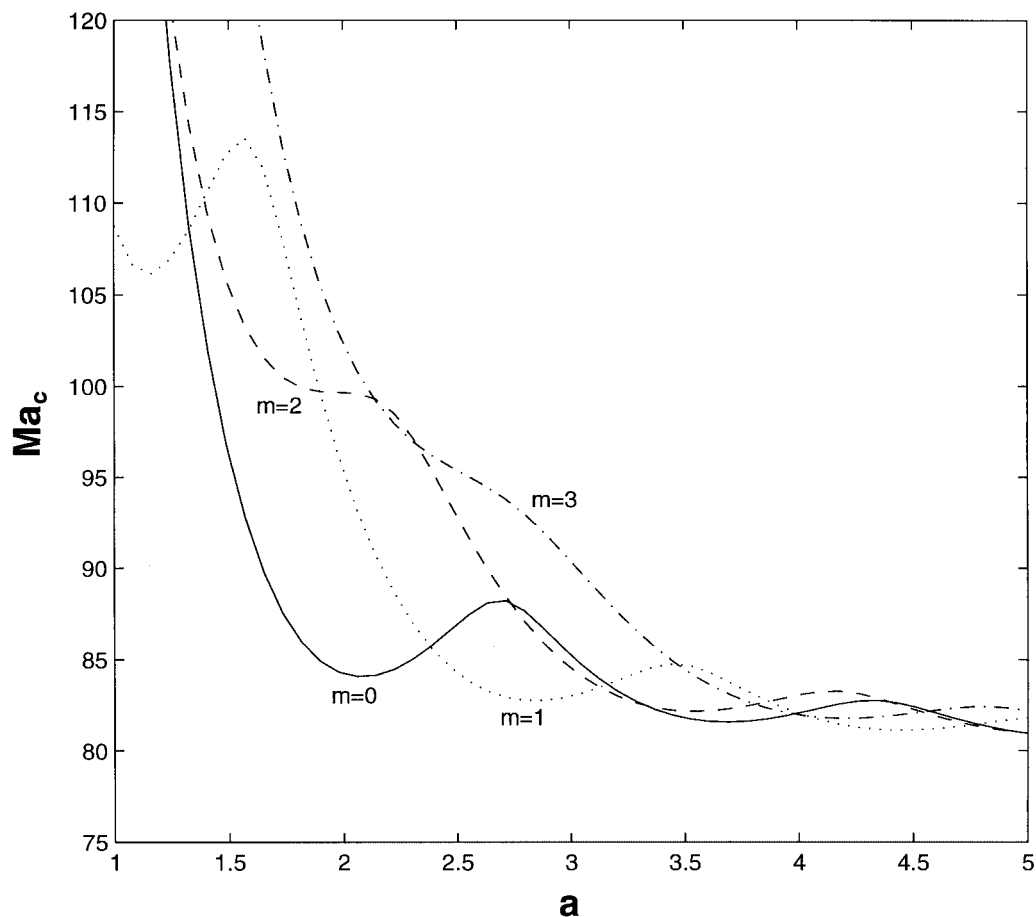


Figure 1. Critical Marangoni number M_c as a function of the aspect ratio a . Curves corresponding to different azimuthal wave numbers m are represented. The Rayleigh and Biot numbers are zero.

it was straightforward to compare the performance of the numerical results in the different formulations. We prove that the results obtained do not depend on those boundary conditions. Sensitivity to those boundary conditions is only observed in the convergence of the method. The best results are obtained in the primitive variables problem, taking as boundary condition for pressure the projection of Navier–Stokes equations at the top boundary and continuity at the bottom. Second, considering the boundary condition with the best convergence we have solved the BM problem in cylindrical geometry. We have compared with previous numerical results and the correspondence is very high. We conclude that the primitive equations with the appropriate boundary conditions for pressure of these thermoconvective problems are accurately solved with our collocation method.

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